Math 251 - Advanced Calculus I, FINAL EXAM SOLUTIONS

1. Consider the sequence of functions

$$f_n(x) = x^n, \quad x \in [0, 1].$$

Note that each function f_n is uniformly continuous, because $[0, 1]$ is compact and any continuous function on a compact set is uniformly continuous.

a) **Give definition of equicontinuity of sequence of functions.**
You have in texts.

b) **Show that the sequence of functions $\{f_n\}$ is not equicontinuous.**
Solution. Let $x_n = 1 - \frac{1}{n}$ and $y_n = 1$. Then $|x_n - y_n| = \frac{1}{n} < \delta$ implies

$$|f_n(x_n) - f_n(y_n)| = |(1 - \frac{1}{n})^n - 1|$$

When $n \to \infty$, $= e^{-1} - 1| = 1 - \frac{1}{e^n} > 0$. Therefore, if we choose $\epsilon > 0$ such that $\epsilon < 1 - \frac{1}{e^n}$ (e.g. $\epsilon = \frac{1-e^{-1}}{2}$) then there exists $n_0 > 0$ such that for all $n \geq n_0$

$$|f_n(x_n) - f_n(y_n)| > \epsilon$$

Thus $\{f_n\}$ is not an equicontinuous family.
c) **Write the Arzela-Ascoli Theorem.**
You have in texts.

2. Show that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n\sqrt{n}} e^{-nx}$$

converges uniformly on $x \geq 0$.

Solution. Let $\phi_n(x) = e^{-nx}$ and $f_n(x) = \frac{\sin(nx)}{n^{3/2}}$. Then $\phi_n(x)$ is decreasing and bounded.
On the other hand,

$$|\frac{\sin(nx)}{n^{3/2}}| \leq \frac{1}{n^{3/2}}$$

By the Weierstrass M-Test, $\sum f_n(x)$ is uniformly convergent.
Therefore, by the Abel's Test, series converges uniformly on $x \geq 0$.

3. Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)\ (x)^n}{n}$$

Solution. By the Ratio test

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \ldots = |x|$$

So, $|x| < 1 \implies -1 < x < 1$.
Check end points:
For $x = -1, \sum \frac{\ln(n)}{n}$ divergent. ($\frac{1}{n} \leq \frac{\ln(n)}{n}$)
For $x = 1, \sum (-1)^n \frac{\ln(n)}{n}$ convergent alternating series, $\frac{\ln(n)}{n}$ decreasing and converge to zero for $n > e$.
Thus interval of convergence is $-1 < x \leq 1$.

4. Consider the following sets

$$A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\}$$

$$B = \{x \in \mathbb{R} : -1 < x \leq 2\} \cap \{x \in \mathbb{R} : x = -\frac{1}{2},\ 0 < x < 3\}$$

a) Sketch their graphs and decide whether A and B are open, bounded, compact, connected?
b) Find interiors, boundaries, closures and accumulation points of A and B.

Justify your answers.

Solution. $B = (0, 2] \cup \{-\frac{1}{2}\}$ is not open, not closed, not compact, not connected. It is bounded.
On the other hand, A is not open, not closed, not compact. It is connected and bounded.

5. (Bonus) Show that $f(x) = 2\sqrt{x} - 5\cos x + \ln(x^2 + 1)$ is **uniformly continuous** on $(1, \infty)$

Solution. For $x \in (1, \infty)$ we have

$$f'(x) = \frac{1}{\sqrt{x}} + 5 \sin x + \frac{2x}{x^2 + 1}$$

Since $|2x| \leq x^2 + 1,\ \sqrt{x} \geq 1$ we have $|f'(x)| \leq 1 + 5 + 1 = 7,\ \forall x \in (1, \infty)$.
Therefore f is uniformly continuous on $(1, \infty)$.