Question 1 (20 points) Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{3^{n+3}}$$

Compute its **sum** for $-2 < x < 4$. Is this series uniformly convergent on $(0,3)$?

Solution. Applying Ratio Test, we easily get $\frac{|x-1|}{3} < 1$ and since it is divergent at the end points we obtain

$$-2 < x < 4$$

as the interval of convergence. Radius of convergence is $R = 3$.
Series uniformly converges on $(0,3)$ because any power series is uniformly convergent in its radius of convergence.

On the other hand, it is a geometric series with $a = \frac{1}{3^3}$ and $r = -\frac{x-1}{3}$.
Therefore

$$Sum = \frac{a}{1-r} = \frac{1}{9(x+2)}$$

Question 2 (20 points) Let $f_n(x) = nxe^{-nx^2}$, $n \in \mathbb{N}$

a) Show that $f_n(x)$ converges pointwise on $[0,1]$ as $n \to \infty$.

b) Show that $f_n(x)$ does not converge uniformly on $[0,1]$ as $n \to \infty$.

c) Calculate

$$\int_0^1 (\lim_{n\to\infty} f_n(x))\, dx$$

and

$$\lim_{n\to\infty} (\int_0^1 f_n(x)\, dx)$$

separately.

Solution. a) $\lim\limits_{n\to\infty} f_n(x) = \ldots = 0$ for every x and $n \in \mathbb{N}$.

b) $f_n'(x) = n[e^{-nx^2}.1 + x.e^{-nx^2}(-2nx)] = 0$ implies $x = \frac{1}{\sqrt{2n}} \in [0,1]$ which easily can be seen that the maximum point. Therefore

$$T_n(x) = Sup_{x\in[0,1]} |f_n(x) - f(x)| = \sqrt{\frac{n}{2e}} \to \infty$$

So it is not uniformly convergent.

c) $\displaystyle\int_0^1 \frac{x}{e^{nx^2}}\, dx = \int_0^n \frac{x}{e^u} . \frac{du}{2nx} = \frac{-1}{2n}(e^{-u}|_0^n) = \frac{-1}{2n}(\frac{1}{e^n} - 1)$ by letting $u = nx^2$.

Therefore

$$\lim_{n\to\infty} (\int_0^1 \frac{nx}{e^{nx^2}}\, dx) = \frac{1}{2}$$

On the other hand

$$\int_0^1 (\lim_{n\to\infty} f_n(x))\, dx = 0$$

Question 3 (30 points)

a) Using the geometric series, find the series expansion of $\dfrac{1}{1+x^2}$

b) Find the Taylor expansion of $f(x) = \arctan(x)$ around $x = 0$ using

$$\arctan(x) = \int \frac{1}{1+x^2}\, dx$$

Why is it valid to exchange summation and integration?

c) Using the result in part (b), show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Solution.

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n, \ \forall x \in (0,1)$$

Replace $-x$ by x^2 we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \ \forall x \in (0,1)$$

Since convergence is uniform then it is valid to exchange summation and integration. Therefore

$$\arctan x = \int \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}, \ \forall x \in (0,1)$$

If we look at the last power series corresponds to $\arctan x$ it is also convergent for $x = 1$. Namely,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

is convergent by the alternating series test.
Therefore we obtain

$$\arctan 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Question 4 (30 points) Determine whether or not the following series converge uniformly for $x \in \mathbb{R}$.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$

Solution. a) If $x \neq 0$ then since it is a geometric series , we get

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 . \sum_{n=0}^{\infty} (\frac{1}{1+x^2})^n = x^2 . \frac{1}{1 - \frac{1}{1+x^2}} = 1 + x^2$$

Therefore

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 1 + x^2, & \text{if } \ x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Limit function is not continuous. So the series does not converge uniformly.

b) You may use several techniques for this question. Let me give an easy one:

Since $2ab \leq a^2 + b^2$ then
$$1 + nx^2 \geq 2\sqrt{n}.|x|,\ \forall x \in \mathbb{R}.$$

So,
$$|\frac{x}{n(1+nx^2)}| \leq \frac{|x|}{2n\ \sqrt{n}\ |x|} = \frac{1}{2n^{\frac{3}{2}}},\ \forall x \in \mathbb{R} \setminus \{0\}.$$

Since $g_n(0) = 0$ this inequality holds $\forall x \in \mathbb{R}$.
Therefore
$$|\frac{x}{n(1+nx^2)}| \leq \frac{1}{2n^{\frac{3}{2}}}$$

By the Weierstrass M-Test, original series is uniformly convergent.