EXERCISES and SOLVED PROBLEMS

1) Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

a) $f_n(x) = x + \frac{x\sin(nx)}{n}$, $x \in [-2,2]$.

b) $f_n(x) = nx(1-x^2)^n$, $x \in [-2,2]$.

2) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^3+x^2}$$

converge uniformly for $x \in [0,\infty)$.

3) Consider the set $D = \{(x,y) : x^2+y^2 < 1\} \cup (\frac{1}{2}, \frac{\sqrt{3}}{2})$.
a) Is D a closed set? Is it an open set? b) Find closure of D, c) Write the set of all accumulation points of D.

4) Let $f_n(x) = \frac{1}{1+n^2x^2}$ and $g_n(x) = nx.(1-x)^n$, $x \in [0,1]$ be given.
a) Show that $(f_n$ is convergent pointwise but not uniformly.
b) Show that (g_n) is convergent pointwise to $g(x) = 0$.

Solution. a) (f_n) is convergent pointwise to $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \leq 1 \end{cases}$. Since f is not continuous, then (f_n) can not be uniformly convergent.

b) Here (g_n) is convergent pointwise to $g(x) = 0$. Because if $x = 0$ or $x = 1$ then $g_n(x) = 0$. Therefore, fix $x \in (0,1)$

$$\lim_{n\to\infty} |\frac{g_{n+1}(x)}{g_n(x)}| = \lim_{n\to\infty} \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} = \lim_{n\to\infty} \frac{n+1}{n}(1-x) = 1-x < 1$$

So $\sum_{n=1}^{\infty} g_n(x)$ converges absolutely by the Ratio Test. Thus

$$\lim_{n\to\infty} g_n(x) = 0$$

5) Consider $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } 0 < x \leq 1 \\ \frac{1}{2}, & \text{if} x = 0 \end{cases}$

Show that $f(x)$ is not continuous on $[0,1]$.

6) Is there a continuous function on $\mathbb{R}$ with range $f(\mathbb{R}) = \mathbb{Q}$? Justify your answer.
Solution. We know that $\mathbb{R}$ is connected but $\mathbb{Q}$ is not. So, by theorem there is no continuous function f with $f(\mathbb{R}) = \mathbb{Q}$.

7) Note that the following theorem is very useful for proving non-uniform continuity.

Theorem (Non-Uniform Continuity Criterion): Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be given. The following are equivalent:
a) f is not uniformly continuous on A.
b) There exists $\epsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim_{n\to\infty}(x_n - u_n) = 0$

and $|f(x_n) - f(u_n)| \geq \epsilon_0, \ \forall n \in \mathbb{N}$.

Using this theorem, show that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$.

Solution. Let $x_n = \frac{1}{\sqrt{n}}, u_n = \frac{1}{\sqrt{n+1}}$. Then

$$|x_n - u_n| = |\frac{1}{\sqrt{n}} - \frac{1}{n+1}| \leq \frac{1}{\sqrt{n}} + \frac{1}{n+1} \to 0$$

but $|f(x_n) - f(u_n)| = |n - (n+1)| = 1, \forall n \in \mathbb{N}$.

8) Show that $f(x) = x^{\frac{1}{3}}$ is a Lipschitz function on $(1, \infty)$.

Solution.

$$|f(x) - f(u)| = |x^{\frac{1}{3}} - u^{\frac{1}{3}}| = \frac{|x-u|}{x^{\frac{2}{3}} + x^{\frac{1}{3}}u^{\frac{1}{3}} + u^{\frac{2}{3}}} \leq \frac{1}{3}|x-u|$$

Therefore f is Lipschitz and so it is uniformly continuous on $(1, \infty)$.

9) Give an example of a function that is uniformly continuous but not a Lipschitz function.

Solution. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 2]$ but not a Lipschitz function.
i) Since it is continuous on the compact interval $[0, 2]$, then it is uniformly continuous.
ii)

$$|\sqrt{x} - \sqrt{u}| = \frac{1}{\sqrt{x} + \sqrt{u}}|x-u|$$

But $\frac{1}{\sqrt{x}+\sqrt{u}}$ is unbounded as $x, u \to 0$.
Therefore it is not Lipschitz.

10) a) Prove that, if f and g are uniformly continuous and bounded on $\mathbb{R}$, then fg is uniformly continuous on $\mathbb{R}$.
b) Show that the conclusion in (a) does not hold if one of the function is unbounded.