SOLUTIONS

MCS 251-MT2 QUESTIONS
December 05, 2012

Name: Signature:

Number:

Q1. Let $f : (a, b) \to \mathbb{R}$ be differentiable and suppose that there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$.Then prove that f is uniformly continuous on (a, b).

Q2. Determine whether the sequence of functions

$$F_n(x) = 2x(1 - x)^n\sqrt{n}, x \in [0, 1]$$

converges uniformly on the indicated interval. Justify your answer.

Q3. Let $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n + x^2}$.
a) Is (f_n) converges uniformly on $[0, 1]$?
b) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$
Describe your answer.

Q4. Determine whether the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n + 2x}}$ converges uniformly on $[0, \infty)$.

Q5. (Bonus) Suppose that $\lim_{n\to\infty} n^{3/2}a_n = 1$ where $a_n \geq 0$ for all n. Does the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converge? Explain your answer .

SOLUTIONS

① Since $f \in D(a,b)$ and $f \in C(a,b) \Rightarrow$ MVT, $\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow \left| \frac{f(b)-f(a)}{b-a} \right| = |f'(c)| \le M \Rightarrow |f(b)-f(a)| \le M.|b-a|$$

25 which gives, f is uniformly continuous on (a,b).

② $F_n(x) = 2x(1-x)^n \sqrt{n}$, $x \in [0,1]$

25 If $0<x<1$, then $\lim\limits_{n\to\infty} F_n(x) = \lim\limits_{n\to\infty} 2x(1-x)^n\sqrt{n} = \lim\limits_{n\to\infty} 2x.\dfrac{\sqrt{n}}{\frac{1}{(1-x)^n}} = 0$

because $\dfrac{1}{1-x} > 1$ and $F_n(0) = F_n(1) = 0$.

$$\therefore \lim_{n\to\infty} F_n(x) = F(x) = 0 \quad \forall x \in [0,1].$$

Now $T_n(x) = \sup\limits_{0\le x\le 1} |F_n(x) - \underbrace{\cancel{F(x)}}_{0}| = \sup\limits_{0\le x\le 1} |F_n(x)|$

But $F_n'(x) = 2\sqrt{n}(1-x)^n + 2x.\sqrt{n}.n(1-x)^{n-1}.(-1) = 0 \Rightarrow \boxed{x = \frac{1}{n+1}}$ or 1.

Max

$$\therefore T_n = \sup_{0\le x\le 1} |F_n(x)| = \text{Max}\left\{F_n(0), F_n(1), F_n\left(\tfrac{1}{n+1}\right)\right\} = 2\sqrt{n}.\frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^n \to 0$$

$\therefore$ Uniformly convergent on $[0,1]$.

③ a) $f_n(x) = \dfrac{\cos nx}{n+x^2}$ Since $0 \le \left|\dfrac{\cos nx}{n+x^2}\right| \le \dfrac{1}{|n+x^2|} \to 0$, $x \in [0,1]$

25 (15+10) $\lim\limits_{n\to\infty}\left(\dfrac{\cos nx}{n+x^2}\right) = 0$ for $(0<x<1)$, for $x=0 \Rightarrow f_n(x) = \frac{1}{n} \to 0$. $\therefore f(x) = 0, \forall x$.

(15) $T_n(x) = \sup\limits_{0\le x\le 1}\left|\dfrac{\cos nx}{n+x^2}\right| \le \sup\limits_{0\le x\le 1}\left|\dfrac{1}{n+x^2}\right| = \dfrac{1}{n} \to 0$ $\therefore$ uniformly converges

b) (10) $\lim\limits_{n\to\infty}\displaystyle\int_0^1 f_n(x)\,dx = \int_0^1 \underbrace{\cancel{\lim_{n\to\infty} f_n(x)}}_{0}\,dx = \boxed{0}$

(4) let $a_n = \dfrac{1}{\sqrt{n+2x}}$, Then for $x \geq 0$, $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$

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$\therefore$ By the alternating series test, $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+2x}}$ converges pointwise on $[0, \infty)$.

Now, $0 \leq T_n = \sup_{x \geq 0} \left| \sum_{k=n+1}^{\infty} \dfrac{(-1)^k}{\sqrt{k+2x}} \right| \leq \sup_{x \geq 0} \dfrac{1}{\sqrt{n+1+2x}} \leq \dfrac{1}{\sqrt{2n+1}} \to 0$

$\therefore \lim_{n\to\infty} T_n = 0$. So, it is uniformly converges.

(5) If $\lim_{n\to\infty} n^{3/2} a_n = 1$ then by the LCT ; let $b_n = \dfrac{1}{n^{3/2}}$

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Since $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{a_n}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} n^{3/2} a_n = 1 \Rightarrow$ both have the same character!

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges $\Rightarrow$ both converges.

$\therefore \sum_{n=1}^{\infty} a_n$ converges.